Outerplanar graphs with positive Lin–Lu–Yau curvature

George Brooks, Fadekemi Osaye, Anna Schenfisch, Zhiyu Wang, and Jing Yu

In this paper, we show that all simple outerplanar graphs G with minimum degree at least 2 and positive Lin–Lu–Yau Ricci curvature on every edge have maximum degree at most 9. Furthermore, if G is maximally outerplanar, then G has at most 10 vertices. Both upper bounds are sharp.

1. Introduction

Ricci curvature plays a crucial role in the geometric analysis of Riemannian manifolds. Roughly speaking, it measures the degree to which the geometry of a metric tensor differs locally from that of a Euclidean space. Curvature in the continuous setting has been studied in depth throughout the past 200 years, and more recent interest has arisen in establishing analogous results in metric spaces. The definition of the (non-combinatorial) Ricci curvature on metric spaces first came from the Bakry and Émery notation [1] who defined the "lower Ricci curvature bound" through the heat semigroup $(P_t)_{t \geq 0}$ on a metric measure space. Ollivier [19] defined the coarse Ricci curvature of metric spaces in terms of how much small balls are closer (in Wasserstein transportation distance) than their centers are. This notion of coarse Ricci curvature on discrete spaces was also made explicit in the Ph.D. thesis of Sammer [21]. The first definition of Ricci curvature on graphs was introduced by Chung and Yau in [4]. To obtain a good log-Sobolev inequality, they defined *Ricci-flatness* in graphs. Later, Lin and Yau [13] gave a generalization of the lower Ricci curvature bound in the framework of graphs. In [12], Lin, Lu, and Yau modified Ollivier's Ricci curvature [19] and defined a new variant of Ricci

arXiv: 2403.04110

curvature on graphs, which does not depend on the idleness of the random walk. For other variants of curvature on discrete spaces, see e.g., [5, 7] and the references therein.

In this paper, we are interested in studying *planar* and *outerplanar* graphs with positive curvature. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Infinite planar graphs are often treated as the discrete version of noncompact simply connected 2-dimensional manifolds. Thus it's natural to consider the 'curvature' of planar graphs. A notion of curvature, previously more extensively studied in graph theory, is the *combinatorial curvature*, which is defined on the vertices of a graph. Given a graph G that is 2-cell embedded in a surface without loops or multiple edges, the *combinatorial curvature* of $v \in V(G)$ is defined as $K(v) := 1 - \frac{d_v}{2} + \sum_{\sigma \in F(v)} \frac{1}{|\sigma|}$, where d_v denotes the degree of v, $F(v)$ is the multiset of faces touching v and $|\sigma|$ is the size of the face σ. A graph G is said to have positive combinatorial curvature everywhere if $\phi(v) > 0$ for all $v \in V(G)$. Higuchi [10] conjectured that if G is a simple connected graph embedded into a 2-sphere with positive combinatorial curvature everywhere and minimum degree $\delta(G) \geq 3$, then G is finite. Higuchi's conjecture was verified by Sun and Yu [22] for cubic planar graphs and resolved by DeVos and Mohar [6]. In particular, DeVos and Mohar showed the following theorem.

Theorem 1 ([6]). *Suppose G is a connected simple graph embedded into a 2-dimensional topological manifold Ω without boundary and G has minimum degree at least 3. If G has positive combinatorial curvature, then it is finite and Ω is homeomorphic to either a 2-sphere or a projective plane. Moreover, if G is not a prism, an antiprism, or one of their projective plane analogues, then $|V(G)| \leq 3444$.*

The minimum possible constants for $|V(G)|$ in Theorem 1 for G embedded in a 2-sphere and a projective plane, respectively, were studied in [20, 17, 3, 23, 18]. In particular, Nicholson and Sneddon [17] gave examples of positively (combinatorially) curved graphs with 208 vertices embedded into a 2-sphere. The upper bound on $|V(G)|$ was recently settled by Ghidelli [9]. Planar graphs with nonnegative combinatorial curvature have also been extensively studied (see [11] and the references within). Tight upper bounds on certain graph classes with positive Lin–Lu–Yau curvature everywhere are also obtained in [8].

Recently, Lu and Wang [15] initiated the study of the order of planar graphs with positive *Lin–Lu–Yau* curvature (*LLY curvature* for short), which

is defined on the vertex pairs of a graph. A graph G is called *positively LLY-curved* if it has positive Lin–Lu–Yau curvature on every vertex pair of G. For brevity of the introduction, we give the definition of the Lin–Lu–Yau curvature in Section 2. In [15], Lu and Wang established an analogue of DeVos and Mohar's result in the context of Lin–Lu–Yau curvature. In particular, they showed the following two theorems.

Theorem 2 ([15])**.** *Let G be a simple positively LLY-curved planar graph G with $\delta(G) \geq 3$. Then the maximum degree $\Delta(G) \leq 17$.*

Theorem 3 ([15])**.** *If G is a simple positively LLY-curved planar graph with minimum degree at least 3, then G is finite. In particular, $|V(G)| \leq 17^{544}$.*

The question of determining a sharp upper bound on the order of a positively LLY-curved planar graph is still open and seems hard. In this paper, we obtain sharp upper bounds on the maximum degree and order of a positively LLY-curved (maximal) outerplanar graph. A graph is *outerplanar* if it admits a planar embedding such that all vertices lie on the outer face. A *maximal outerplanar* graph is an edge maximal outerplanar graph. We first show a sharp upper bound on the maximum degree of a positively LLY-curved outerplanar graph.

Theorem 4. *Let G be a simple positively LLY-curved outerplanar graph with $\delta(G) \geq 2$. Then $\Delta(G) \leq 9$ and the upper bound is sharp.*

Using Theorem 4 and additional degree constraints shown in Section 4, we obtain a sharp upper bound on the order of positively LLY-curved maximal outerplanar graphs.

Theorem 5. *Let G be a positively LLY-curved maximal outerplanar graph. Then $|V(G)| \leq 10$ and the upper bound is sharp.*

The upper bounds in Theorem 4 and Theorem 5 are both achieved by the fan graph on 10 vertices (see Figure 1). We remark that very recently, Liu, Lu and Wang [14] extended Theorem 5 to the class of all outerplanar graphs (not necessarily maximal) with minimum degree 2.

For convenience, in the rest of the paper, we simply say a graph G is positively curved if it is positively LLY-curved, and we simply denote $\kappa_{LLY}(x, y)$ by $\kappa(x, y)$.

Notation and Terminology. Given a graph G and $x \in V(G)$, we use $N(x)$ to denote the neighborhood of x, i.e., $N(x) = \{y \in V(G) : xy \in E(G)\}$. Let d_x denote the degree of x, i.e., $d_x = |N(x)|$. Let $N[x] := N(x) \cup \{x\}$.
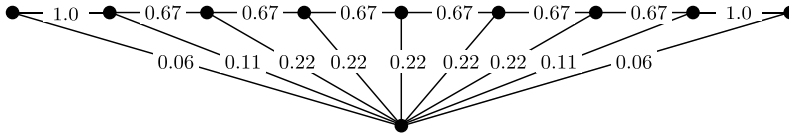
Figure 1: Fan graph on 10 vertices with curvature of each edge computed by SageMath.

Given $x, y \in V(G)$, let $N(x, y)$ denote the common neighborhood of x and y, i.e., $N(x, y) := N(x) \cap N(y)$. Moreover, given $S \subseteq V(G)$, define $N(S) := \{x \in V(G) : xs \in E(G) \text{ for some } s \in S\}$ and $E(x, S) := \{xs \in E(G) : s \in S\}$. Throughout the paper, for $xy \in E(G)$, we will consider the *local configuration* $H_{xy} \subseteq G$, the induced subgraph of G defined by $H_{xy} = G[N[x] \cup N[y]]$. For $x, y \in V(G)$, we use $d(x, y)$ to denote the number of edges in a shortest path between x and y. Given an outerplanar graph G and an edge e, we call e an *exterior edge* if it lies on the boundary of the outer face of G, and we call e an *interior edge* otherwise.

2. Lin–Lu–Yau Ricci curvature

In this section, we define the Lin–Lu–Yau Ricci curvature. We closely follow the notation of [15, 19]. Let m_1 and m_2 be two probability distributions on $V(G)$. That is, for $i = 1, 2$, we have $m_i : V(G) \to [0, 1]$ such that $\sum_{x \in V(G)} m_i(x) = 1$. A *coupling* between m_1 and m_2, is a map $A : V(G) \times V(G) \to [0, 1]$ with finite support such that $\sum_{y \in V(G)} A(x, y) = m_1(x)$ and $\sum_{x \in V(G)} A(x, y) = m_2(y)$. The *transportation distance* between m_1 and m_2 is defined as

$$(1) \qquad W(m_1, m_2) = \inf_A \sum_{x, y \in V(G)} A(x, y)\, d(x, y),$$

where the infimum is taken over all couplings A between m_1 and m_2. By the duality theorem of a linear optimization problem, the transportation distance can also be written as

$$(2) \qquad W(m_1, m_2) = \sup_{f \in Lip(1)} \sum_{x \in V(G)} f(x)\, (m_1(x) - m_2(x)),$$

where the supremum is taken over all 1-Lipschitz functions f. A random walk m on a graph $G = (V, E)$ is defined as a family of probability measures

$\{m_v(\cdot)\}_{v \in V(G)}$ such that $m_v(u) = 0$ for all $uv \notin E(G)$. It follows that $m_v(u) \geq 0$ for all $v, u \in V(G)$ and $\sum_{u \in N[v]} m_v(u) = 1$. For $v \in V(G)$, consider the α-lazy random walk $\{m_v^\alpha\}_{v \in V(G)}$ defined as

$$m_v^\alpha(u) = \begin{cases} \alpha & \text{if } u = v, \\ \frac{1-\alpha}{d_v} & \text{if } u \in N(v), \\ 0 & \text{otherwise.} \end{cases}$$

In [12], Lin, Lu and Yau defined the Ricci curvature of graphs based on the α-lazy random walk as α goes to 1. More precisely, for any $x \neq y \in V$, they defined the α-Ricci-curvature $\kappa_\alpha(x, y)$ to be

$$(3) \qquad \kappa_\alpha(x, y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x, y)}$$

and the Lin–Lu–Yau Ricci curvature κ_{LLY} of the vertex pair (x, y) in G to be

$$(4) \qquad \kappa_{\mathrm{LLY}}(x, y) = \lim_{\alpha \to 1} \frac{\kappa_\alpha(x, y)}{(1 - \alpha)}.$$

Recall that a graph G is called positively LLY-curved if it has positive Lin–Lu–Yau curvature on every vertex pair of G. It was shown [12] that if $\kappa_{\mathrm{LLY}}(x, y) \geq \kappa_0$ for every edge $xy \in E(G)$, then $\kappa_{\mathrm{LLY}}(x, y) \geq \kappa_0$ for any pair of vertices (x, y).

Recently, Münch and Wojciechowski [16] gave a limit-free formulation of the Lin–Lu–Yau Ricci curvature using *graph Laplacian*. Given a graph $G = (V, E)$, the graph Laplacian Δ is defined as

$$(5) \qquad \Delta f(v) = \frac{1}{d_v} \sum_{u \in N(v)} (f(u) - f(v)).$$

The Lin–Lu–Yau curvature can be alternatively expressed [16] as

$$(6) \qquad \kappa_{LLY}(x, y) = \inf_{\substack{f \in Lip(1), \\ \nabla_{yx} f = 1}} \nabla_{xy} \Delta f,$$

where $\nabla_{vu} f = \frac{f(v) - f(u)}{d(v, u)}$ is the *gradient* function. Following Lemma 2.2 of [2], it suffices to optimize over all integer-valued 1-Lipschitz functions f. An integer-valued 1-Lipschitz function $f : V(G) \to \mathbb{Z}$ that attains the infimum in Equation (6) is called *optimal for* (x, y).

3. Proof of Theorem [4]

In this section, we show a sharp upper bound on the maximum degree of a positively curved outerplanar graph. We first recall the following lemma of Lu and Wang [15].

Lemma 1 ([15]). *Let G be a simple positively curved planar graph and $xy \in E(G)$ be an arbitrary edge with $d_x \leq d_y$. Suppose $S \subseteq N(x) \setminus \{y\}$, $|S| = s$, and $|S \cap N(y)| = k$. Then*

$$|N(S) \cap N(y)| > \frac{s}{d_x} d_y - (k + 1 + |N(x,y)|) + |N(S) \cap N(x,y)|.$$

When $S = N(x) \setminus \{y\}$ in the equation above, we have $s = d_x - 1$ and $k = |N(x,y)|$. We then obtain the following immediate corollary of Lemma [1].

Corollary 1. *Let G be a simple positively curved planar graph and $xy \in E(G)$ be an arbitrary edge with $d_x \leq d_y$. Let $S = N(x) \setminus \{y\}$. Then*

$$|N(S) \cap N(y)| > \left(1 - \frac{1}{d_x}\right) d_y - (2|N(x,y)| + 1) + |N(S) \cap N(x,y)|.$$

Proof of Theorem [4]. Suppose, towards a contradiction, that $\Delta(G) \geq 10$. Let $x, y \in V(G)$ such that $d_y = \Delta(G)$ and $xy \in E(G)$ is an exterior edge. Then $d_x \geq \delta(G) \geq 2$. Let $S = N(x) \setminus \{y\}$ and let $T := N(S) \cap N(y)$. Since xy is an exterior edge, the common neighborhood $N(x,y)$ may either be empty, or contain one vertex. We consider these cases separately.

Case 1: Suppose $|N(x,y)| = 0$. Using Corollary [1],

$$
\begin{aligned}
|T| &> \left(1 - \frac{1}{d_x}\right) d_y - 1 + |N(S) \cap N(x,y)| \\
&\geq \frac{1}{2} \cdot 10 - 1 + |N(S) \cap N(x,y)| \\
&\geq 4.
\end{aligned}
$$

Hence $|T| \geq 5$. However, note that contracting all edges in $E(x, S)$ yields a $K_{2,3}$ minor in G, contradicting the outerplanarity of G.

Case 2: Suppose $|N(x,y)| = 1$ and let z be the only common neighbor in $N(x,y)$. If $d_x = 2$, then $|S| = |N(x) \setminus \{y\}| = 1$ and

$$|T| > \left(1 - \frac{1}{d_x}\right) d_y - 3 + |N(S) \cap N(x,y)|$$

$$\geq \frac{1}{2} \cdot 10 - 3 + |N(S) \cap N(x,y)|$$
$$\geq 2.$$

Hence, $|T| \geq 3$. Therefore, the only vertex in S (which is z) is adjacent to at least 3 neighbors of y, yielding a $K_{2,3}$ minor and contradicting the outerplanarity of G. If $d_x \geq 3$, then

$$|T| > \left(1 - \frac{1}{d_x}\right) d_y - 3 + |N(S) \cap N(x,y)|$$
$$\geq \frac{2}{3} \cdot 10 - 3 + |N(S) \cap N(x,y)|$$
$$\geq \frac{11}{3}.$$

Hence, $|T| \geq 4$. Since $d_y = \Delta(G) \geq 10$ and $d_x \geq 3$, the sets $N(y) \setminus \{x,z\}$ and $S \setminus \{z\}$ must be nonempty. For any $v \in N(y) \setminus \{x,z\}$ and $w \in S \setminus \{z\}$, if $vw \in E(G)$, then either z lies in the interior of the cycle $wxyv$ which contradicts that G is outerplanar, or xy lies in the interior of the cycle $wxyv$ which contradicts that xy is an exterior edge. Thus, we have that for any $v \in N(y) \setminus \{x,z\}$ and $w \in S \setminus \{z\}$, $vw \notin E(G)$.

Therefore, any $v \in T \setminus \{x,z\}$ must be adjacent to z. Since $|T \setminus \{x,z\}| \geq 2$ and $x \in N(z)$, z is adjacent to at least three vertices in $N(y)$, yielding a $K_{2,3}$ minor together with y and $N(z,y)$, contradicting the outerplanarity of G. $\square$

4. Exact curvature via local configuration

In this section, we characterize the possible degree pairs of an edge with positive Lin–Lu–Yau curvature in a maximal outerplanar graph by determining the optimal 1-Lipschitz functions for every edge $xy \in E(G)$. Recall that
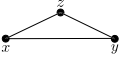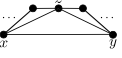
$$(7) \qquad \kappa(x,y) = \inf_{\substack{f \in Lip(1), \\ \nabla_{yx} f = 1}} \nabla_{xy} \Delta f, \quad \text{where } \nabla_{xy} f = \frac{f(x) - f(y)}{d(x,y)}.$$

Moreover, recall that by Lemma 2.2 of [2], it suffices to consider integer-valued optimal functions f_0. Thus, since $\nabla_{yx} f_0 = 1$, we may assume without loss of generality that $f_0(x) = 0$ and $f_0(y) = 1$. Then, due to the 1-Lipschitz condition, $\text{Im}(f_0|_{N(x)}) \subseteq \{-1,0,1\}$ and $\text{Im}(f_0|_{N(y)}) \subseteq \{0,1,2\}$ where Im denotes the image of a function.

Lemma 2. *Let G be a maximal outerplanar graph, and suppose $xy \in E(G)$ is an exterior edge such that $d_x \leq d_y$. Then, $\kappa_{LLY}(x, y) > 0$ if and only if*

$$(d_x, d_y) \in \{(2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 3), (3, 4)\}.$$

Table 1: Degree pairs for positively curved exterior edges xy and local configuration H_{xy}

H_{xy}	δ_{xz}	d_x	δ_{yz}	d_y	$\kappa(x, y)$	Positive degree pairs (d_x, d_y)
	0	2	0	2	$3/d_x + 4/d_y - 2$	$\{(2,2)\}$
	0	2	1	≥ 3	$\begin{cases} 4/d_x + 3/d_y - 2 & \text{if } d_y < 2d_x \\ 3/d_x + 5/d_y - 2 & \text{if } 2d_x \leq d_y \end{cases}$	$\{(2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9)\}$
	1	≥ 3	1	$\geq d_x$	$3/d_x + 5/d_y - 2$	$\{(3,3), (3,4)\}$

Proof of Lemma 2. Let xy be an exterior edge of G. We will deduce optimal functions f_0 for the edge xy in all possible local configurations around xy, that is, f_0 such that

$$(8) \qquad \kappa(x, y) = \inf_{\substack{f \in \text{Lip}(1), \\ \nabla_{yx} f = 1}} \nabla_{xy} \Delta f = \nabla_{xy} \Delta f_0.$$

Without loss of generality, we assign $f_0(x) = 0$ and $f_0(y) = 1$. Since xy is an exterior edge and G is edge maximal, $N(x) \cap N(y) = \{z\}$ for some $z \in V(G)$. Then, since $f_0 \in \text{Lip}(1)$, we must have $f_0(z) \in \{0, 1\}$. Let $\delta_{xz} = |N(x, z) \setminus \{y\}|$ and $\delta_{yz} = |N(y, z) \setminus \{x\}|$. Note that, since G is outerplanar, $\delta_{xz}, \delta_{yz} \in \{0, 1\}$. If $\delta_{xz} = 1$, let $N(x, z) \setminus \{y\} = \{v_{xz}\}$. Similarly, if $\delta_{yz} = 1$, let $N(y, z) \setminus \{x\} = \{v_{yz}\}$. Since $f_0(x) = 0$ and $f_0(y) = 1$, we then obtain that

$$\nabla_{xy} \Delta f_0 = \frac{1}{d_x} \sum_{v \in N(x)} (f_0(v) - f_0(x)) - \frac{1}{d_y} \sum_{v \in N(y)} (f_0(v) - f_0(y))$$

$$(9) \qquad = \frac{1}{d_x} \sum_{v \in N(x)} f_0(v) + \frac{1}{d_y} \sum_{v \in N(y)} (1 - f_0(v)).$$

Observe that since G is outerplanar, xy is an exterior edge and $\{z\} = N(x, y)$, we have that the distance from any vertex in $N(x) \setminus \{v_{xz}, y, z\}$ to

any vertex in $N(y) \setminus \{v_{yz}, x, z\}$ is 3. Since f_0 is an optimal function for xy, we then obtain that $f_0(v) = -1$ for all $v \in N(x) \setminus \{v_{xz}, y, z\}$ and $f_0(v) = 2$ for all $v \in N(y) \setminus \{v_{yz}, x, z\}$. Observe that if $\delta_{xz} = 1$, i.e., v_{xz} exists, then by the 1-Lipschitz and integer-valued property of f_0, we have that $f_0(v_{xz}) \in \{f_0(z), f_0(z) - 1\}$. Now since f_0 is optimal, it is easy to see that $f_0(v_{xz}) = f_0(z) - 1$ so that $\Delta f_0(x)$ is minimal in (8). Similarly, if $\delta_{yz} = 1$, then $f_0(v_{yz}) = f_0(z) + 1$. Equation (9) can then be re-written as

$$\nabla_{xy} \Delta f_0 = \frac{1}{d_x} \left[\delta_{xz} f_0(v_{xz}) + f_0(z) + 1 - (d_x - 2 - \delta_{xz}) \right]$$

$$+ \frac{1}{d_y} \left[\delta_{yz}(1 - f_0(v_{yz})) + (1 - f_0(z)) + 1 - (d_y - 2 - \delta_{yz}) \right]$$

$$= \frac{1}{d_x} \left[\delta_{xz}(f_0(z) - 1) + f_0(z) + 1 - (d_x - 2 - \delta_{xz}) \right]$$

$$+ \frac{1}{d_y} \left[-\delta_{yz} f_0(z) + (1 - f_0(z)) + 1 - (d_y - 2 - \delta_{yz}) \right]$$

$$(10) \qquad = \left(\frac{1 + \delta_{xz}}{d_x} - \frac{1 + \delta_{yz}}{d_y} \right) f_0(z) + \frac{3}{d_x} + \frac{4 + \delta_{yz}}{d_y} - 2.$$

Note that $\frac{1 + \delta_{xz}}{d_x} - \frac{1 + \delta_{yz}}{d_y} < 0$ if and only if $\delta_{xz} = 0, \delta_{yz} = 1$ and $d_y < 2d_x$.

Case 1: $\frac{1 + \delta_{xz}}{d_x} - \frac{1 + \delta_{yz}}{d_y} < 0$. Then it is clear that $\nabla_{xy} \Delta f_0$ is minimized when $f_0(z) = 1$. Moreover, since in this case $\delta_{xz} = 0$, f_0 still satisfies the 1-Lipschitz condition.

Case 2: $\frac{1 + \delta_{xz}}{d_x} - \frac{1 + \delta_{yz}}{d_y} \geq 0$. Then it is clear that $\nabla_{xy} \Delta f_0$ is minimized when $f_0(z) = 0$. Suppose $\delta_{xz} = 1$ and $\delta_{yz} = 0$, then $d_x \geq 3$. Since G is maximally outerplanar, yz must be an exterior edge, which implies $d_y = 2$, a contradiction of $d_x \leq d_y$. Hence, it follows that $(\delta_{xz}, \delta_{yz}) \in \{(0, 0), (0, 1), (1, 1)\}$. We list each local configuration based on δ_{xz} and δ_{yz} in Table 1 and characterize the possible degree pairs when $\kappa(x, y) > 0$. It is not hard to check that f_0 can be made 1-Lipschitz in each case. □

Lemma 3. *Let G be a maximal outerplanar graph, and suppose $xy \in E(G)$ is an interior edge such that $d_x \leq d_y$. A complete characterization of degree pairs (d_x, d_y), depending upon the local configuration, such that $\kappa_{LLY}(x, y) > 0$ is given by Table 2.*

Proof of Lemma 3. Let xy be an interior edge of G. We will deduce optimal functions f_0 for the edge xy in all possible local configurations around xy.

Table 2: Positive degree pairs for interior edges xy based on the local configuration H_{xy}

H_{xy}	δ_{xw}	δ_{xz}	d_x	δ_{yw}	δ_{yz}	d_y	$\kappa(x,y)$	Positive degree pairs (d_x,d_y)
	0	0	3	0	0	3	$4/d_x+6/d_y-2$	$\{(3,3)\}$
	0	0	3	0	1	≥ 4	$\begin{cases}5/d_x+5/d_y-2 & \text{if } d_y<2d_x\\4/d_x+7/d_y-2 & \text{if } 2d_x\leq d_y\end{cases}$	$\{(3,4),(3,5),(3,6),(3,7),(3,8),(3,9)\}$
	0	0	3	1	1	≥ 5	$\begin{cases}6/d_x+4/d_y-2 & \text{if } d_y<2d_x\\4/d_x+8/d_y-2 & \text{if } 2d_x\leq d_y\end{cases}$	$\{(3,5),(3,6),(3,7),(3,8),(3,9)\}$
	0	1	≥ 4	0	1	$\geq d_x$	$4/d_x+7/d_y-2$	$\{(4,4),(4,5),(4,6),(5,5)\}$
	0	1	≥ 4	1	0	$\geq d_x$	$\begin{cases}5/d_x+5/d_y-2 & \text{if } d_y<2d_x\\4/d_x+7/d_y-2 & \text{if } 2d_x\leq d_y\end{cases}$	$\{(4,4),(4,5),(4,6)\}$
	0	1	≥ 4	1	1	$\geq \max\{5,d_x\}$	$\begin{cases}5/d_x+6/d_y-2 & \text{if } d_y<2d_x\\4/d_x+8/d_y-2 & \text{if } 2d_x\leq d_y\end{cases}$	$\{(4,5),(4,6),(4,7),(5,5)\}$
	1	1	≥ 5	0	1	$\geq d_x$	$4/d_x+7/d_y-2$	$\{(5,5)\}$
	1	1	≥ 5	1	1	$\geq d_x$	$4/d_x+8/d_y-2$	$\{(5,5),(5,6)\}$

Similar to before, without loss of generality, we assume that $f_0(x) = 0$ and $f_0(y) = 1$.

Since xy is an interior edge and G is edge maximal, $N(x)\cap N(y) = \{w,z\}$ for some $w,z \in V(G)$ and by the 1-Lipschitz condition we have $f_0(w), f_0(z) \in \{0,1\}$. Similar to before, Equation (9) still holds, i.e.,

$$\nabla_{xy}\Delta f_0 = \frac{1}{d_x}\sum_{v\in N(x)}(f_0(v) - f_0(x)) - \frac{1}{d_y}\sum_{v\in N(y)}(f_0(v) - f_0(y))$$

$$(11) \qquad = \frac{1}{d_x}\sum_{v\in N(x)}f_0(v) + \frac{1}{d_y}\sum_{v\in N(y)}(1 - f_0(v)).$$

Define $\delta_{xw} = |N(x,w)\setminus\{y\}|$, $\delta_{xz} = |N(x,z)\setminus\{y\}|$, $\delta_{yw} = |N(y,w)\setminus\{x\}|$ and $\delta_{yz} = |N(y,z)\setminus\{x\}|$. Since G is outerplanar, $\delta_{xw},\delta_{xz},\delta_{yw},\delta_{yz} \in \{0,1\}$. For $a \in \{x,y\}$ and $b \in \{w,z\}$, if $\delta_{ab} = 1$, then let $N(a,b)\setminus\{x,y\} = \{v_{ab}\}$.

Observe that in (11), $\nabla_{xy}\Delta f_0$ is minimized when $f_0(v)$ is as small as

possible for $v \in N(x)$ and as big as possible for $v \in N(y)$ while keeping f_0 1-Lipschitz. Note that since G is outerplanar, $\{w, x\}$ separates v_{xw} (if it exists) with vertices in $N(y) \backslash \{x, w\}$. Similar to Lemma 2, if $\delta_{xw} = 1$, then $f_0(v_{xw}) = f_0(w) - 1$ due to the optimality of f_0. Similarly, $f_0(v_{xz}) = f_0(z) - 1$, $f_0(v_{yw}) = f_0(w) + 1$, $f_0(v_{yz}) = f_0(z) + 1$ if v_{xz}, v_{yw}, v_{yz} exist respectively.

Moreover, note that the distance from any vertex in $N(x) \setminus \{v_{xw}, w, y, z, v_{xz}\}$ to any vertex $N(y) \setminus \{v_{yw}, w, x, z, v_{yz}\}$ is 3. Therefore, since f_0 is optimal, we obtain that $f_0(v) = -1$ for all $v \in N(x) \setminus \{v_{xw}, w, y, z, v_{xz}\}$ and $f_0(v) = 2$ for all $v \in N(y) \setminus \{v_{yw}, w, x, z, v_{yz}\}$. Then Equation (11) becomes

$$\nabla_{xy} \Delta f_0 = \frac{1}{d_x} \left[\delta_{xw}(f_0(w) - 1) + f_0(w) + 1 + f_0(z) + \delta_{xz}(f_0(z) - 1) \right.$$

$$\left. - (d_x - 3 - \delta_{xw} - \delta_{xz}) \right] + \frac{1}{d_y} \left[-\delta_{yw} f_0(w) + (1 - f_0(w)) + 1 \right.$$

$$\left. + (1 - f_0(z)) - \delta_{yz} f_0(z) - (d_y - 3 - \delta_{yw} - \delta_{yz}) \right]$$

$$= \left(\frac{1 + \delta_{xw}}{d_x} - \frac{1 + \delta_{yw}}{d_y} \right) f_0(w) + \left(\frac{1 + \delta_{xz}}{d_x} - \frac{1 + \delta_{yz}}{d_y} \right) f_0(z)$$

$$+ \frac{4}{d_x} + \frac{6 + \delta_{yw} + \delta_{yz}}{d_y} - 2.$$

Note that $\frac{1+\delta_{xw}}{d_x} - \frac{1+\delta_{yw}}{d_y} < 0$ if and only if $\delta_{xw} = 0, \delta_{yw} = 1$ and $d_y < 2d_x$. In that case, since f_0 is optimal, it must be the case that $f_0(w) = 1$. On the other hand, if $\frac{1+\delta_{xw}}{d_x} - \frac{1+\delta_{yw}}{d_y}$ is nonnegative, $\nabla_{xy}\Delta f_0$ is minimized when $f_0(w) = 0$.

Similarly, note that $\frac{1+\delta_{xz}}{d_x} - \frac{1+\delta_{yz}}{d_y} < 0$ if and only if $\delta_{xz} = 0, \delta_{yz} = 1$ and $d_y < 2d_x$. In that case, it must happen that $f_0(z) = 1$ as before. On the other hand, if $\frac{1+\delta_{xz}}{d_x} - \frac{1+\delta_{yz}}{d_y} \geq 0$, we may assume $f_0(z) = 0$. Note that the case when $\delta_{xw} + \delta_{xz} \geq 1$ and $\delta_{yw} = \delta_{yz} = 0$ is impossible since $d_x \leq d_y$. We list each local configuration (up to symmetry) for $\delta_{xw}, \delta_{xz}, \delta_{yw}$ and δ_{yz} in Table 2 and compute the possible degree pairs when $\kappa(x, y) > 0$. □

5. Bounding the order

In this section, we bound the order of a positively curved maximal outerplanar graph G. Note that since G is maximally outerplanar, if $|V(G)| \geq 3$, then G is connected and $\delta(G) \geq 2$. Given two graphs A and B, let the *join* of A and B, denoted by $A \vee B$, be the graph obtained from $A \cup B$ by connecting each

vertex in A with each vertex in B. We begin with the following structural lemma.

Lemma 4. *Let G be a maximal outerplanar graph and suppose $x \in V(G)$. Then, $G[N[x]] \cong K_1 \vee P_{d_x}$.*

Proof. Since G is outerplanar, $G[N(x)]$ must be a disjoint union of paths. Moreover, since G is edge maximal, $G[N(x)]$ must be a single path (otherwise we could add an edge to merge two paths into one without violating the outerplanarity condition). Thus $G[N[x]] \cong K_1 \vee P_{d_x}$. □

Proof of Theorem 5. We will show Theorem 5 by induction on $|V(G)|$. For the base case, we consider $|V(G)| = 11$. Using SageMath, we verify that all 228 maximal outerplanar graphs on 11 vertices are non-positively curved.[1]

Now consider G, a maximal outerplanar graph on $n \geq 12$ vertices. Assume, towards a contradiction, that G is positively curved. Let $w \in V(G)$ such that $d_w = 2$ (such w exists since every maximal outerplanar graph contains a vertex of degree 2). Note that $G - w$ is maximally outerplanar. It follows from the induction hypothesis that $G - w$ is not positively curved, thus there exists some $xy \in E(G)$ with $\kappa(x, y) \leq 0$. Without loss of generality, assume that $d_x \leq d_y$. We write d_v for the degree of v in G, and we write $\tilde{d}_v$ for the degree of v in $G - w$. Since xy is non-positively curved in $G - w$, Lemmas 2 and 3 exactly characterize the possible values of $(\tilde{d}_x, \tilde{d}_y)$; we call degree pairs corresponding to non-positively curved edges *bad* degree pairs and degree pairs corresponding to positively curved edges *good* degree pairs.

First, suppose $w \notin N(x) \cup N(y)$, so that $(d_x, d_y) = (\tilde{d}_x, \tilde{d}_y)$. Since $(\tilde{d}_x, \tilde{d}_y)$ is a bad degree pair and the degrees of x and y are the same in G and $G - w$, we have that (d_x, d_y) is a bad degree pair, a contradiction. Next, suppose $w \in N(x) \setminus N(y)$ so that $(d_x, d_y) = (\tilde{d}_x + 1, \tilde{d}_y)$. Notice that since $w \in N(x) \setminus N(y)$, xy is an exterior (or interior) edge in G if and only if it is an exterior (respectively, interior) edge in $G - w$. Examining Table 1 and Table 2, we see that, given a degree pair (a, b) that is bad for exterior (or interior) edges, the degree pair $(a + 1, b)$ is also bad for exterior (respectively, interior) edges. Thus, (d_x, d_y) is a bad degree pair in G, contradicting our assumption that G is positively curved. The case when $w \in N(y) \setminus N(x)$ is similar.

The remainder of the proof addresses the final case, where $w \in N(x, y)$, so that $(d_x, d_y) = (\tilde{d}_x + 1, \tilde{d}_y + 1)$. Note that in this case, xy has to be exterior

---

[1]The SageMath code used to verify is available at: https://github.com/ghbrooks28/outerplanar-LLY/.

Figure 2: H_{xy} in Case 1 and 2.

edge of $G - w$. Since xy is non-positively curved in $G - w$, we obtain from Lemma 2 that

$$(12) \qquad \begin{aligned} (\widetilde{d}_x, \widetilde{d}_y) \notin \{&(2,2),(2,3),(2,4),(2,5),(2,6), \\ &(2,7),(2,8),(2,9),(3,3),(3,4)\}. \end{aligned}$$

Note that by Theorem 4, $\Delta(G) \le 9$. It follows that $\max\{\widetilde{d}_y, \widetilde{d}_x\} \le 8$. By Equation (12), it suffices to consider xy such that $\widetilde{d}_x \ge 3$ and $\widetilde{d}_y \ge 4$, where $\widetilde{d}_y > 4$ if $\widetilde{d}_x = 3$. Then by Lemma 3, $(d_x, d_y) = (\widetilde{d}_x + 1, \widetilde{d}_y + 1)$ is a bad degree pair (and so G would be non-positively curved) for all values except $(d_x, d_y) \in \{(4,6),(5,5),(4,7),(5,6)\}$. However, note that when $(d_x, d_y) = (4,7)$ or $(5,6)$, $\max\{\delta_{xw}, \delta_{yw}\} = 1$ (see Table 2 in Lemma 3), which contradicts that $d_w = 2$. Hence we are left with two cases when $(d_x, d_y) \in \{(4,6),(5,5)\}$. We will examine the local configuration H_{xy} below.

Case 1: Suppose $(d_x, d_y) = (4,6)$. Let $C \subseteq G$ be the outer face cycle in G. Using Lemma 4, we have the local configuration H_{xy}, labeled as shown in Figure 2 (left). Let $x' \in N_C(x) \setminus \{w\}$ and $y', v, u \in N_C(y) \setminus \{w\}$ such that $y'vuzxw$ is the path induced by $N(y)$. Since $d_y = 6$, yy' is an exterior edge, and yz, yu, yv are interior edges. Note that G is assumed to be positively curved. Thus the degree pairs of yy', yz, yu and yv all have to be good. By Lemma 2 and Lemma 3, we have $d_{y'} = 2$, $d_z \in \{4,5\}$, and $d_u, d_v \in \{3,4,5\}$. Since $d_x = 4$, and xx' is an exterior edge, $d_{x'} \in \{2,3\}$.

We first claim that $d_{x'} = 2$. Otherwise, if $d_{x'} = 3$, then there exists $x'' \in V(G) \setminus \{x\}$ such that $x'' \in N(x', z)$, which implies $d_z \ge 5$. By Lemma 2 and Lemma 3, zu must then be an interior edge, which implies that $d_z \ge 6$. But now (d_y, d_z) is a bad degree pair, contradicting that yz is positively curved. Hence $d_{x'} = 2$.

But now if we repeat the argument by considering $G - x'$ instead of $G - w$, we obtain contradictions unless $(d_x, d_z) \in \{(4,6),(5,5)\}$, which is impossible in this case since $d_x = 4$ and $d_z < 6$.

Case 2: Suppose $(d_x, d_y) = (5,5)$. Let $C \subseteq G$ be the outer face cycle. Let $x', u \in N_C(x) \setminus \{w\}$ such that $x'uzyw$ is the path induced by $N(x)$ (see

Figure 2 (right)). Since $d_x = 5$, and xx' is an exterior edge, it follows from
Lemma 2 that $d_{x'} = 2$. Now if we repeat the argument by considering $G - x'$
instead of $G - w$, we obtain contradictions unless $(d_x, d_u) = (5, 5)$. However,
this is impossible since by the configurations in Lemma 3, if $(d_x, d_u) = (5, 5)$,
then $d_{x'} > 2$, giving a contradiction.

This completes the proof of Theorem 5.   □

Acknowledgments

References

[1] D. Bakry and M. Émery, Diffusions hypercontractives, Séminaire de
    probabilités, XIX, 1983/84, 177–206, Lecture Notes in Math. 1123,
    Springer, Berlin, 1985. MR0889476

[2] B. Bhattacharya and S. Mukherjee, Exact and asymptotic results on
    coarse Ricci curvature of graphs, *Discrete Math.*, **338(1)** (2015), 23–42.
    MR3273535

[3] B. Chen and G. Chen, Gauss–Bonnet formula, finiteness condition,
    and characterizations of graphs embedded in surfaces, *Graphs Combin.*,
    24(3) (2008), 159–183. MR2410938

[4] F. Chung and S.-T. Yau, Logarithmic Harnack inequalities, *Math. Res.
    Lett.* **3** (1996), 793–812. MR1426537

[5] K. Devriendt and R. Lambiotte, Discrete curvature on graphs from the
    effective resistance, *J. Phys.: Complex.*, **3(2)** (2022), 025008.

[6] M. DeVos and B. Mohar, An analogue of the Descartes–Euler formula
    for infinite graphs and Higuchi's conjecture, *Trans. Amer. Math. Soc.*,
    359(7) (2007), 3287–3300. MR2299456

[7] R. Forman, Bochner's method for cell complexes and combinatorial Ricci curvature, *Discrete Comput. Geom.*, **29** (2003), 323–374. MR1961004

[8] E. Gamlath, X. Liu, L. Lu and X. Yuan, A tight bound on $\{C_3, C_5\}$-free connected graphs with positive Lin–Lu–Yau Ricci curvature, *arXiv:2312.16593*.

[9] L. Ghidelli, On the largest planar graphs with everywhere positive combinatorial curvature, *J. Combin. Theory Ser. B.*, **158(2)** (2023), 226–263. MR4494559

[10] Y. Higuchi, Combinatorial curvature for planar graphs, *J. Graph Theory* **38(4)** (2001), 220–229. MR1864922

[11] B. Hua and Y. Su, The set of vertices with positive curvature in a planar graph with nonnegative curvature, *Adv. Math.* **343** (2019), 789–820. MR3891984

[12] Y. Lin, L. Lu and S.-T. Yau, Ricci curvature of graphs, *Tohoku Math. J.* **63** (2011) 605–627. MR2872958

[13] Y. Lin and S.-T. Yau, Ricci curvature and eigenvalue estimate on locally finite graphs, *Math. Res. Lett.* **17** (2010), 345—358. MR2644381

[14] X. Liu, L. Lu and Z. Wang, On the size of outerplanar graphs with positive Lin-Lu-Yau Ricci curvature, *arXiv:2409.13666*.

[15] L. Lu and Z. Wang, On the size of planar graphs with positive Lin–Lu–Yau Ricci curvature, *arXiv:2010.03716*.

[16] F. Münch and R. Wojciechowski, Ollivier Ricci curvature for general graph Laplacians: Heat equation, Laplacian comparison, non-explosion and diameter bounds, *Adv. Math.*, **356** (2019). MR3998765

[17] R. Nicholson and J. Sneddon, New graphs with thinly spread positive combinatorial curvature *New Zealand J. Math.*, **41** (2011), 39–43. MR2836763

[18] B.-G. Oh, On the number of vertices of positively curved planar graphs, *Discrete Math.*, **340(6)** (2017), 1300–1310. MR3624614

[19] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, *J. Funct. Anal.* **256** (2009), 810–864. MR2484937

[20] T. Réti, E. Bitay and Z. Kosztolányi, On the polyhedral graphs with positive combinatorial curvature, *Acta Polytech. Hungar.*, **2(2)** (2005), 19–37.

[21] M.D. Sammer. Aspects of mass transportation in discrete concentration inequalities. PhD thesis, Georgia Institute of Technology, 2005. MR2707115

[22] L. Sun and X. Yu, Positively curved cubic plane graphs are finite, *J. Graph Theory* **47(4)** (2004), 241–274. MR2096789

[23] L. Zhang, A result on combinatorial curvature for embedded graphs on a surface, *Discrete Math.*, **308(24)** (2008), 6588–6595. MR2466966

GEORGE BROOKS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SC
USA
E-mail address: ghbrooks@email.sc.edu

FADEKEMI OSAYE
TROY UNIVERSITY
TROY, AL
USA
E-mail address: fosaye@troy.edu

ANNA SCHENFISCH
EINDHOVEN UNIVERSITY OF TECHNOLOGY
EINDHOVEN
NETHERLANDS
E-mail address: a.k.schenfisch@tue.nl

ZHIYU WANG
LOUISIANA STATE UNIVERSITY
BATON ROUGE, LA
USA
E-mail address: zhiyuw@lsu.edu

JING YU
FUDAN UNIVERSITY
SHANGHAI
CHINA
E-mail address: jyu@fudan.edu.cn